Journal of Engineering Mathematics 25: 265–285, 1991.

A mathematical model for the initial stages of fluid impact in the presence of a cushioning fluid layer

S.K. WILSON
Mathematical Institute, University of Oxford, 24–29 St. Giles', Oxford OX2 3LB, England; present address: School of Mathematics, University of East Anglia, Norwich NR4 7TJ, England

Received and accepted 9 August 1990

Abstract. In this paper we formulate a mathematical model for initial stages of the impact of a rigid body onto a fluid in the presence of a cushioning fluid layer between them. Some numerical and analytical solutions are obtained in appropriate asymptotic limits, and numerical solutions are obtained to the full leading order problem. Finally, we compare our model with the work of other authors.

1. Introduction

Owing to the many practical applications of fluid impact problems, such as in ship dynamics and the wave loading of offshore structures, considerable mathematical and engineering effort has been expended obtaining approximate solutions to idealised fluid impact problems. A variety of methods have been employed, ranging from the entirely empirical to those employing the techniques of matched asymptotic expansions, but most ignore the presence of the air between the fluid and the impacting body and assume that the free surface is planar at the moment of impact. Of the latter type Cointe and Armand [3], Cointe [4] and Howison et al. [9] have analysed the special case when the *deadrise angle* (the angle between the tangent to the body and the horizontal) is small, i.e. when the impacting body is almost flat. Figure 1 shows a typical set of experimental pressure histories recorded during drop tests using a roughly parabolic body conducted by Nethercote et al. [14], and the corresponding theoretical predictions from an asymptotic theory based on small deadrise angle reproduced from Howison et al. [9]. Although the incompressibile theory over-predicts the size of the maximum pressure peak during the early stages, the theoretical predictions are in good agreement with the experimental results away from the keel. However, at the keel the incompressible theory predicts an infinite pressure whilst rapid pressure oscillations are observed experimentally. The explanation for these oscillations is the presence of a pocket of air trapped between the body and the fluid surface. It is the formation of this 'cushion' of air that we shall investigate in this paper and Fig. 2 shows the typical two-dimensional geometry we shall consider. In general, as the body approaches the fluid the free surface will be deformed by the pressure in the air, and will therefore not be planar at the moment of impact.

Since the pioneering work by Von Kármán [16] and Wagner [17] many authors have investigated the impact of a solid body onto a fluid. The presence of the unknown free surface of the fluid makes the problem a formidable non-linear one and, even for a rigid body impacting onto a quiescent half-space of inviscid and incompressible fluid without a cushioning air layer, no exact solutions are known. Despite this a great deal of progress has

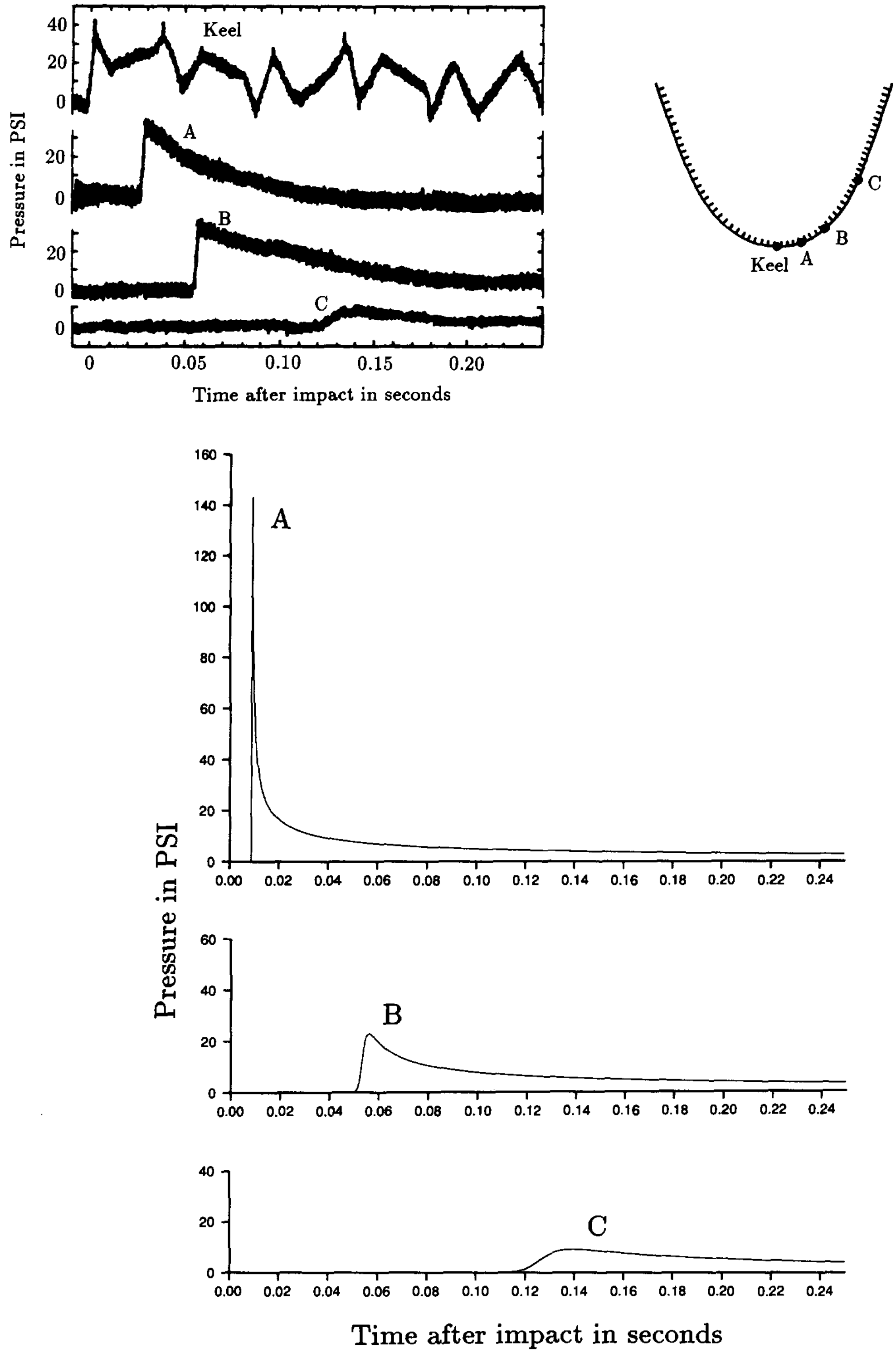

Fig. 1. Comparison of experimental and theoretical pressure histories. Experimental data from Nethercote et al. [14] for a roughly parabolic body, 12 ft wide impacting at 20 ft/s. Theoretical predictions reproduced from Howison et al. [9] based on a small deadrise angle asymptotic theory. At the keel the theory predicts an infinite pressure and the oscillations observed experimentally are probably due to air entrapment.

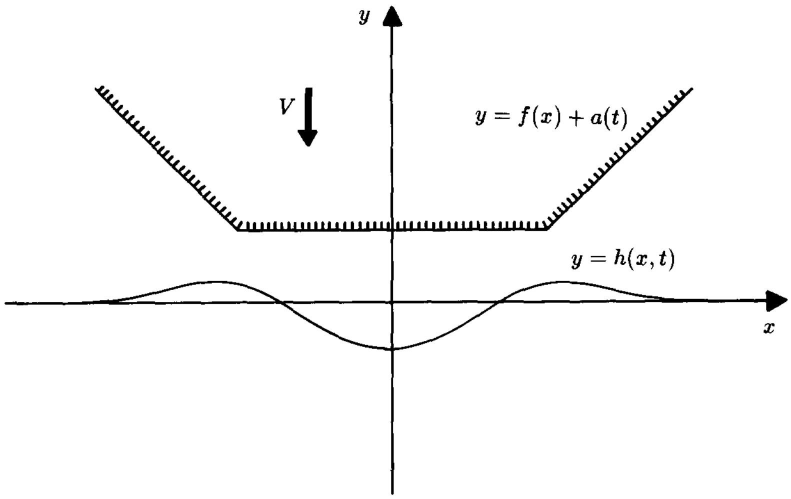

Fig. 2. Air cushioning geometry.

been made, and a number of review articles, such as that by Moran [13] and more recently Greenhow [7] and Korobkin and Pukhnachov [10] have appeared.

In the present work we concentrate on the problem of air cushioning during impact, which has been investigated analytically and numerically by a number of authors.

Verhagen [15] used a simple one-dimensional model for the compressible flow of air in the narrowing gap between a finite length flat plate and a fluid surface, which he solved numerically. Motivated by the well-known theory applying to the steady flow of compressible fluid in converging and diverging channels, he assumed that as the air velocity reached the local sound speed in the throat formed between the body and the rising water surface, the flow would choke and that thereafter the air speed in the throat would be equal to the sound speed. The flow in this regime was calculated until the instant that the body first touched the water. Then a model for the trapped air pocket, in which it was assumed that the air pressure was a function of time only, was used to predict the pressure on the body. Despite incorporating a number of approximations, and introducing an arbitrary smoothing factor into the pressure distribution, the calculations were shown to be in good agreement with a set of experimental measurements made using a light-weight model. Lewison and Maclean [11] and Lewison [12] reported an extensive series of drop tests using flat plates, and compared the results with numerical solutions to an approximate one-dimensional model for the flow of the air which incorporated empirical assumptions about the pressure. The computed solutions were in quantitative agreement with the experiments, but displayed a marked sensitivity to initial conditions and overestimated the pressures by roughly a factor of two. A series of experiments with models of ship sections showed that adding flanges to the keel encouraged air entrapment and reduced the measured impact pressures. Apparently working without reference to the earlier work, Asryan [1] described two different models for the air layer beneath a flat plate. He first modelled the air as an inviscid and incompressible fluid. In order to make progress he assumed that the deflection of the free surface was small compared to the thickness of the air gap, and this approximation led to a single non-linear integral equation for the pressure in the air. The second approach was to model the air as a viscous and compressible fluid, and led to the equations of a viscous squeeze film and the

Reynolds equation for the flow in a channel with moving walls. Again, to make progress he assumed that the free surface deflection was small, and this led to a single quasi-linear partial differential equation for the pressure. Approximate solutions to these two equations were obtained numerically, and in both cases estimates for the time of first contact between the plate and the water were calculated.

Driscoll and Lloyd [5] reported a series of drop tests using flat-bottomed wedges of varying keel size and deadrise angle, and made measurements of the speed and magnitude of the maximum pressure pulse. First contact was usually made at the edge of the keel, and for larger keels a smaller, inward travelling pressure pulse, caused as the pocket of trapped air collapsed, was recorded. Chuang [2] described a series of drop tests performed with flat-bottomed and small deadrise angle wedges. His experiments indicated that only the flat bottomed and 1° deadrise angle wedges entrapped a significant quantity of air, and he derived a sequence of empirical corrections to Wagner's [17] simple formula for the maximum pressure in the absence of air. Hagiwara and Yuhara [8] conducted experiments to measure the impact forces and resulting stresses on a number of one third scale ship bow models. They too only observed the effects of air entrapment for wedge angles less than about 3°. Eroshin et al. [6] reported a series of drop test experiments onto compressible fluids using flat bodies in the presence of various types of cushioning fluid layers, and found them to be in good agreement with numerical solutions to a simple one-dimensional model.

2. Problem formulation

The most likely explanation for the pressure oscillations measured on the keel of a roughly parabolic body during a fluid impact, shown in Fig. 1, is that the pressure in the air between the solid and the liquid is not negligible, and so the free surface is deformed before the impact occurs and a cushioning pocket of air is trapped. In order to discuss this mechanism, we derive a model which incorporates the air flow before an impact.

We consider the impact of a two-dimensional, rigid, symmetric body onto a quiescent half-space of inviscid and incompressible fluid, and take cartesian axes (x, y) with the y-axis vertically upwards and the x-axis along the undisturbed free surface. Working in dimensional variables, the position of the body is denoted by $y = f(x) + a(t)$, where the body has profile $y = f(x)$ with $f(0) = 0$, and $a(t)$ is the distance of the body at $x = 0$ from the level of the undisturbed free surface. Initially air fills the space between the body and the undisturbed water surface, and we neglect the effects of gravity and surface tension at the free surface. We shall model the air as an inviscid and incompressible fluid, but note that the compressibility of the air will become significant whenever the speed of the air becomes comparable with the local sound speed, which is most likely to occur as the air-gap narrows just before the body first touches the water surface. For simplicity we have referred to the two fluids in the model as air and water, but obviously the model is more general, and can be applied to any two immiscible, inviscid and incompressible fluids.

Since the flows in the air and in the water are both initially irrotational, Kelvin's theorem applies to them, and so they will remain irrotational throughout the motion and we can define appropriate velocity potentials, ϕ_1 in the air and ϕ_2 in the water. It is sometimes more convenient to work directly with the x- and y-components of velocity, and so we denote these by $u_i(x, y, t) = \partial\phi_i/\partial x$ and $v_i(x, y, t) = \partial\phi_i/\partial y$ respectively, where $i = 1$ for quantities in the air and $i = 2$ for those in the water. The governing equations in both fluids are Euler's equations;

$$\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} = 0 , \tag{1}$$

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} + v_i \frac{\partial u_i}{\partial y} = -\frac{1}{\rho_i} \frac{\partial p_i}{\partial x} , \tag{2}$$

$$\frac{\partial v_i}{\partial t} + u_i \frac{\partial v_i}{\partial x} + v_i \frac{\partial v_i}{\partial y} = -\frac{1}{\rho_i} \frac{\partial p_i}{\partial y} , \tag{3}$$

for $i = 1, 2$. Extending our notation, $p_1(x, y, t)$, $p_2(x, y, t)$ are the pressure in the air and the water and ρ_1, ρ_2 denote the density of the air and of the water respectively.

At the surface of the body we require continuity of normal velocity between the body and the air,

$$u_1 f'(x) - v_1 = -a'(t) \quad \text{on} \quad y = f(x) + a(t) . \tag{4}$$

At the free boundary between the air and the water, denoted by $y = h(x, t)$, the kinematic condition applied to each fluid gives

$$\frac{\partial h}{\partial t} + u_1 \frac{\partial h}{\partial x} - v_1 = 0 \quad \text{and} \quad \frac{\partial h}{\partial t} + u_2 \frac{\partial h}{\partial x} - v_2 = 0 \quad \text{on} \quad y = h(x, t) . \tag{5}$$

In the absence of gravity and surface tension, the air and water pressures must be equal at the free surface, and so from Bernoulli's equation we have

$$\rho_1\left(\frac{\partial \phi_1}{\partial t} + \frac{1}{2} |\nabla\phi_1|^2\right) = \rho_2\left(\frac{\partial \phi_2}{\partial t} + \frac{1}{2} |\nabla\phi_2|^2\right) \quad \text{on} \quad y = h(x, t) . \tag{6}$$

Appropriate initial conditions are that the air and the water are at rest and that the free surface is planar. The problem has the far-field condition

$$|\nabla\phi_i| \to 0 \quad \text{as} \quad (x^2 + y^2)^{1/2} \to \infty \quad \text{for} \quad i = 1, 2 . \tag{7}$$

3. Non-dimensionalization

First we non-dimensionalize the problem by scaling the variables with the typical horizontal length, l, and typical air-gap thickness, λ. On substituting the new scaled variables into the equations and boundary conditions, we will find that two non-dimensional groups of parameters arise naturally. These are the aspect ratio of the gap, which we denote by

$$\varepsilon = \frac{\lambda}{l} ,$$

and the ratio of the density of air to the density of water, which we denote by

$$\delta = \frac{\rho_1}{\rho_2} \simeq 10^{-3} .$$

The experiments of Lewison [12] suggest that the water surface only responds to the approaching body when it is very close to it, and so we will investigate solutions to our model when both these parameters are small, viz. $\varepsilon \ll 1$ and $\delta \ll 1$.

We scale the vertical velocity of the air with V, a typical vertical speed of the body, and from the mass conservation condition (1) the typical horizontal speed of the air is V/ε. The free surface evaluation scales with $\delta\lambda$, and from the Euler equations (2, 3), we see that the pressure in the air is of order $\rho_1 V^2/\varepsilon^2$. The length scale in the water is l, but the velocity scale, denoted by U, has to be determined. The scale of the water pressure is obtained by balancing the pressure term in the Euler equation (2) with either the inertia term, which is of order U^2/l, or the unsteady term, which is of order UV/l. Choosing the former gives a pressure scale of $\rho_2 U^2$. Since the pressure is continuous across the free boundary, the pressures in the air and in the water must be comparable, and this determines U to be $\delta^{1/2}V/\varepsilon$. However, the ratio of the inertia term to the unsteady term is now $\delta^{1/2} \ll 1$, meaning that the unsteady term dominates the right hand side of the equation, which contradicts our original choice. If instead we balance the pressure term with the unsteady term then the pressure scale is $\rho_2 UV/\varepsilon$, and matching the pressures across the free boundary determines the appropriate value of U to be $V\delta/\varepsilon$, which is consistent.

3.1. The water problem

In the water problem the non-dimensionalization takes the form

$$x^* = \frac{x}{l}, \qquad y^* = \frac{y}{l}, \qquad t^* = \frac{Vt}{l},$$

with

$$u_2^* = \frac{u_2}{U}, \qquad v_2^* = \frac{v_2}{U}, \qquad \phi_2^*(x^*, y^*, t^*) = \frac{1}{Ul}\,\phi_2(x, y, t),$$

and

$$p_2^*(x^*, y^*, t^*) = \frac{\varepsilon^2}{\rho_1 V^2}\,p_2(x, y, t), \qquad h^*(x^*, t^*) = \frac{1}{\delta\lambda}\,h(x, t).$$

Dropping the starred notation for dimensionless quantities, the governing equations in the water are

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0, \tag{8}$$

$$\frac{\partial u_2}{\partial t} + \delta\left[u_2\frac{\partial u_2}{\partial x} + v_2\frac{\partial u_2}{\partial y}\right] = -\frac{\partial p_2}{\partial x}, \tag{9}$$

$$\frac{\partial v_2}{\partial t} + \delta\left[u_2\frac{\partial v_2}{\partial x} + v_2\frac{\partial v_2}{\partial y}\right] = -\frac{\partial p_2}{\partial y}, \tag{10}$$

and the statement that the flow is irrotational, $\nabla \times \mathbf{u}_2 \equiv 0$, takes the form

$$\frac{\partial u_2}{\partial y} - \frac{\partial v_2}{\partial x} = 0. \tag{11}$$

The boundary condition on the free surface is

$$\frac{\partial h}{\partial t} + \delta u_2 \frac{\partial h}{\partial x} - v_2 = 0 \quad \text{on} \quad y = \delta h(x, t)\,, \tag{12}$$

and from Bernoulli's equation the pressure in the water is

$$p_2 = -\left[\frac{\partial \phi_2}{\partial t} + \frac{\delta}{2}(u_2^2 + v_2^2)^{1/2}\right]. \tag{13}$$

The far-field condition is

$$|\nabla \phi_2| \to 0 \quad \text{as} \quad (x^2 + y^2)^{1/2} \to \infty\,. \tag{14}$$

3.2. The air problem

In the air problem the non-dimensionalization takes the form

$$x^* = \frac{x}{l}\,, \qquad y^* = \frac{y}{\lambda}\,, \qquad t^* = \frac{Vt}{l}\,,$$

with

$$u_1^* = \frac{u_1 \lambda}{Vl}\,, \qquad v_1^* = \frac{v_1}{V}\,, \qquad \phi_1^*(x^*, y^*, t^*) = \frac{1}{Vl}\,\phi_1(x, y, t)\,,$$

and

$$p_1^*(x^*, y^*, t^*) = \frac{\varepsilon^2}{\rho_1 V^2}\,p_1(x, y, t)\,, \qquad h^*(x^*, t^*) = \frac{1}{\delta\lambda}\,h(x, t)\,,$$

$$f^*(x^*) = \frac{1}{\lambda}\,f(x)\,, \qquad a^*(t^*) = \frac{1}{\lambda}\,a(t)\,.$$

Dropping the starred notation for dimensionless quantities, the air problem is given by the equations

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0\,, \tag{15}$$

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} = -\frac{\partial p_1}{\partial x}\,, \tag{16}$$

$$\varepsilon^2\left[\frac{\partial v_1}{\partial t} + u_1 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_1}{\partial y}\right] = -\frac{\partial p_1}{\partial y}\,, \tag{17}$$

and the statement that the flow is irrotational, $\nabla \times \mathbf{u}_1 \equiv 0$, takes the form

$$\frac{\partial u_1}{\partial x} - \varepsilon^2 \frac{\partial v_1}{\partial y} = 0\,. \tag{18}$$

The boundary conditions on the body and the free surface are

$$u_1 f'(x) - v_1 = -a'(t) \quad \text{on} \quad y = f(x) + a(t)\,, \tag{19}$$

$$\frac{\delta}{\varepsilon}\left[\frac{\partial h}{\partial t} + u_1 \frac{\partial h}{\partial x}\right] - v_1 = 0 \quad \text{on} \quad y = \frac{\delta}{\varepsilon} h(x, t)\,, \tag{20}$$

and from Bernoulli's equation the pressure in the air is

$$p_1 = -\left[\frac{\partial \phi_1}{\partial t} + \frac{1}{2}(u_1^2 + \varepsilon^2 v_1^2)^{1/2}\right]. \tag{21}$$

The far-field condition is

$$|\nabla\phi_1| \to 0 \quad \text{as} \quad (x^2 + y^2)^{1/2} \to \infty\,. \tag{22}$$

4. The leading order water problem

If we seek a solution for ϕ_2 (and hence u_2 and v_2) as an asymptotic series in δ, in the form

$$\phi_2 = (\phi_2)_0 + \delta(\phi_2)_1 + O(\delta^2)\,,$$

then, dropping the clumsy subscript notation and dealing with leading order quantities unless otherwise stated, the leading order problem is given by the acoustic equations;

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0\,, \tag{23}$$

$$\frac{\partial u_2}{\partial t} = -\frac{\partial p_2}{\partial x}\,, \tag{24}$$

$$\frac{\partial v_2}{\partial t} = -\frac{\partial p_2}{\partial y}\,, \tag{25}$$

with the condition that the flow is irrotational

$$\frac{\partial u_2}{\partial y} - \frac{\partial v_2}{\partial x} = 0\,. \tag{26}$$

The boundary conditions on the free surface are

$$\frac{\partial h}{\partial t} = v_2\,, \qquad p_1 = p_2 \quad \text{on} \quad y = 0\,. \tag{27}$$

If we regard $h(x, t)$ as being determined from the air problem then it is convenient to treat the problem as a boundary value problem for the velocity potential. Since, to leading order,

$$\frac{\partial \phi_2}{\partial y} = \frac{\partial h}{\partial t} \quad \text{on} \quad y = 0\,,$$

the solution for $\partial\phi_2/\partial y$, harmonic in $y \leqslant 0$ and satisfying the boundary condition and the far-field condition (14) is obtained by using the appropriate Green's function;

$$\frac{\partial \phi_2}{\partial y}(x, y, t) = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{\partial h}{\partial t}(\xi, t)\,\frac{d\xi}{(\xi - x)^2 + y^2}\,. \tag{28}$$

Hence, the velocity potential ϕ_2 is given by

$$\phi_2(x, y, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} -\frac{\partial h}{\partial t}(\xi, t) \log|(\xi - x)^2 + y^2| \, d\xi \,. \tag{29}$$

Furthermore,

$$\frac{\partial \phi_2}{\partial x}(x, 0, t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\partial h}{\partial t}(\xi, t) \frac{d\xi}{\xi - x} = \mathscr{H}\left(\frac{\partial h}{\partial t}\right), \tag{30}$$

where $\mathscr{H}(\cdot)$ denotes a *Hilbert Transform*. From equation (24) we deduce that on $y = 0$

$$-\frac{\partial p_2}{\partial x}(x, 0, t) = \frac{\partial}{\partial t} \mathscr{H}\left(\frac{\partial h}{\partial t}\right),$$

and therefore the surface pressure is determined in terms of the free surface elevation to be

$$\frac{\partial p_2}{\partial x}(x, 0, t) = -\mathscr{H}\left(\frac{\partial^2 h}{\partial t^2}\right). \tag{31}$$

5. The leading order air problem

The solution of the air problem is dependent on the ratio δ/ε, and so we introduce the new parameter, ϑ, defined by

$$\vartheta = \frac{\delta}{\varepsilon}\,.$$

If we seek solutions for u_1 and v_1 as an asymptotic series in ε^2, in the form

$$u_1 = (u_1)_0 + \varepsilon^2 (u_1)_1 + O(\varepsilon^4)\,, \qquad v_1 = (v_1)_0 + \varepsilon^2 (v_1)_1 + O(\varepsilon^4)\,,$$

and again drop the subscripts, then the leading order problem is given by the equations

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0\,, \tag{32}$$

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} = -\frac{\partial p_1}{\partial x}\,, \tag{33}$$

$$0 = -\frac{\partial p_1}{\partial y}\,, \tag{34}$$

with the statement that the flow is irrotational

$$\frac{\partial u_1}{\partial y} = 0\,. \tag{35}$$

The boundary condition on the body is

$$u_1 f'(x) - v_1 = -a'(t) \quad \text{on} \quad y = f(x) + a(t)\,, \tag{36}$$

and those on the free surface are

$$\vartheta\left[\frac{\partial h}{\partial t} + u_1 \frac{\partial h}{\partial x}\right] - v_1 = 0\,, \qquad p_1 = p_2 \quad \text{on} \quad y = \vartheta h(x, t)\,. \tag{37}$$

The geometry of the leading order air problem is shown in Fig. 3. From equation (35), we deduce that u_1 is a function of x and t only, and so we can integrate equation (32) with respect to y to obtain

$$v_1 = -\frac{\partial u_1}{\partial x}\, y + \alpha(x, t)\,,$$

where $\alpha(x, t)$ is an unknown function of x and t. Substituting into the boundary conditions (36) and (37a) and eliminating $\alpha(x, t)$ gives

$$\frac{\partial}{\partial t}\{\vartheta h(x, t) - a(t) - f(x)\} + \frac{\partial}{\partial x}[\{\vartheta h(x, t) - a(t) - f(x)\}u_1] = 0\,. \tag{38}$$

Since $\vartheta h(x, t) - a(t) - f(x)$ is the thickness of the air-gap, this is just a statement of conservation of mass in the air. From equation (33)

$$-\frac{\partial p_1}{\partial x} = \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x}\,,$$

and so we deduce that the air pressure is also a function of x and t only. Using equation (31) we can eliminate the pressure and obtain an integral equation relating $h(x, t)$ and $u_1(x, t)$, namely,

$$\mathcal{H}\left(\frac{\partial^2 h}{\partial t^2}\right) = \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x}\,. \tag{39}$$

Equations (38) and (39) are a coupled pair of integral and differential equations for the leading order free surface elevation, $h(x, t)$, and the leading order horizontal air velocity, $u_1(x, t)$. Once they have been solved, the solution in the water can be evaluated by substituting $h(x, t)$ into equation (31) and the problem therefore reduces to solving this coupled pair of equations.

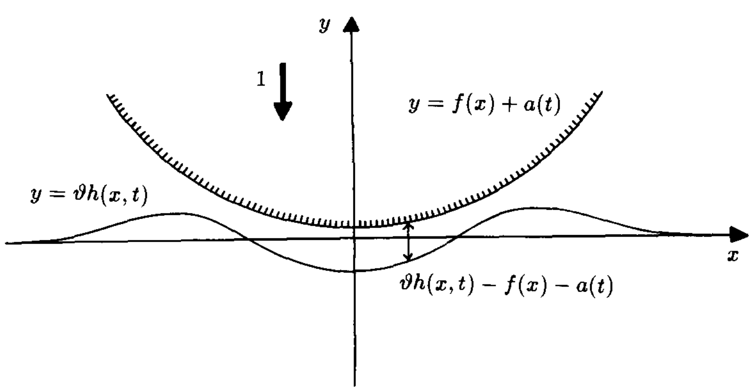

Fig. 3. Geometry of the leading order air problem.

6. Asymptotic solutions

The coupled equations (38) and (39) are analytically intractable, but some progress can be made by seeking solutions in appropriate asymptotic limits before resorting to numerical calculations. For clarity, we drop the subscript one and hereafter denote the leading order horizontal air velocity simply by $u(x, t)$.

6.1. *Small time asymptotic behaviour*

We investigate the small time behaviour of the evolution from an initial state given by $u(x, T) = U_0(x)$, $h(x, T) = H_0(x)$, $a(T) = A_0$ at $t = T$, without restriction on ϑ. We choose the origin of time to be at T and expand $a(t)$ as a power series in time t for $t \ll 1$ in the form

$$a(t) = \sum_{n=0}^{\infty} A_n t^n ,$$

where the A_n for $n = 0, 1, 2, \ldots$ are constants, and seek solutions for $h(x, t)$ and $u(x, t)$ also as power series in t for $t \ll 1$ in the forms

$$u(x, t) = \sum_{n=0}^{\infty} U_n(x) t^n , \qquad h(x, t) = \sum_{n=0}^{\infty} H_n(x) t^n ,$$

where the functions $U_n(x)$ and $H_n(x)$ are to be determined. Substituting these expressions into equation (38) and equating coefficients of t, we obtain the leading order terms

$$O(1) \qquad \vartheta H_1(x) - A_1 + \frac{\partial}{\partial x}[\{\vartheta H_0(x) - (A_0 + f(x))\}U_0(x)] = 0 , \tag{40a}$$

$$O(t) \qquad 2(\vartheta H_2(x) - A_2) + \frac{\partial}{\partial x}[\{\vartheta H_1(x) - A_1\}U_0(x)] + \frac{\partial}{\partial x}[\{\vartheta H_0(x) - (A_0 + f(x))\}U_1(x)] = 0 , \tag{40b}$$

$$O(t^2) \qquad 6(\vartheta H_3(x) - A_3) + \frac{\partial}{\partial x}[\{\vartheta H_1(x) - A_1\}U_1(x)] + \frac{\partial}{\partial x}[\{\vartheta H_0(x) - (A_0 + f(x))\}U_2(x) + \{\vartheta H_2(x) - A_2\}U_0(x)] = 0 , \tag{40c}$$

and from equation (39) the leading order terms

$$O(1) \qquad \mathcal{H}(2H_2(x)) = U_1(x) + U_0(x)\frac{\partial U_0}{\partial x} , \tag{41a}$$

$$O(t) \qquad \mathcal{H}(6H_3(x)) = 2U_2(x) + \frac{\partial}{\partial x}(U_0(x)U_1(x)) . \tag{41b}$$

We note that equation (40a) is a single equation involving three unknown quantities, $H_0(x)$, $H_1(x)$ and $U_0(x)$. However, two of these may be determined from initial conditions at $t = 0$, and taking the next order terms from equations (40) and (41) introduces two equations for two new unknowns at each order. In this way all the terms in the expansions of $u(x, t)$ and $h(x, t)$ can, in principle, be calculated.

Case (a). If we impose the initial conditions

$$h(x, 0) = 0 \quad \text{and} \quad u(x, 0) = 0\,,$$

then $H_0(x) \equiv U_0(x) \equiv 0$, and from equation (40a)

$$H_1(x) = -\frac{A_1}{\vartheta}\,. \tag{42}$$

Substituting these expressions into equation (41a) and inverting the Hilbert transform gives

$$H_2(x) = -\frac{1}{2}\,\mathscr{H}(U_1)\,. \tag{43}$$

Integrating equation (40b) once with respect to x and imposing the symmetry condition $u(0, t) = 0$ for all t, then yields a singular integral equation for $U_1(x)$, the leading order term in the expansion of the velocity, namely

$$U_1(x) = g(x) + \frac{\vartheta}{A_0 + f(x)}\,\frac{1}{\pi}\int_{-\infty}^{+\infty} U_1(\xi)\log\left|1 - \frac{x}{\xi}\right| \mathrm{d}\xi\,, \tag{44}$$

where the function $g(x)$ is given by

$$g(x) = -\frac{2A_2 x}{A_0 + f(x)}\,. \tag{45}$$

Case (b). If we impose the initial conditions

$$h(x, 0) = 0 \quad \text{and} \quad \frac{\partial h}{\partial t}(x, 0) = 0\,,$$

then $H_0(x) \equiv H_1(x) \equiv 0$. Integrating equation (40a) gives a simple expression for the leading order velocity, viz.

$$U_0(x) = -\frac{A_1}{A_0 + f(x)}\,, \tag{46}$$

and equation (40b) yields a formula for the first non-zero term in the free surface elevation,

$$H_2(x) = \frac{1}{2\vartheta}\left[2A_2 + \frac{\partial}{\partial x}\left[\{A_0 + f(x)\}U_1(x) + A_1 U_0(x)\right]\right]. \tag{47}$$

Substituting these expressions into equation (41a), inverting the Hilbert transform and integrating with respect to x yields a singular integral equation for $U_1(x)$,

$$U_1(x) = g(x) + \frac{\vartheta}{A_0 + f(x)}\,\frac{1}{\pi}\int_{-\infty}^{+\infty} U_1(\xi)\log\left|1 - \frac{x}{\xi}\right| \mathrm{d}\xi\,, \tag{48}$$

where the function $g(x)$ is given by

$$g(x) = \frac{A_1^2 x}{(A_0 + f(x))^2} - \frac{2A_2 x}{A_0 + f(x)} + \frac{A_1^2\vartheta}{A_0 + f(x)}\,\frac{1}{\pi}\int_{-\infty}^{+\infty} \xi\left\{\frac{A_0 + f(\xi) - \xi f'(\xi)}{(A_0 + f(\xi))^3}\right\}\log\left|1 - \frac{x}{\xi}\right| \mathrm{d}\xi\,. \tag{49}$$

Both singular integral equations (44) and (48) can, with care, be solved numerically by a method of successive approximations. An initial guess for $U_1(x)$ is substituted into the right hand side of the equation, and the resulting integration performed numerically. The result is a revised approximation for $U_1(x)$, and this procedure is repeated until the desired convergence is obtained.

Figure 4 shows the first term in the expansion of the horizontal air velocity and the two leading order non-zero terms in the expansion of the free surface elevation for a body with profile $y = 3x^6$, subject to the initial conditions (a) with $A_0 = 1$, $A_1 = -1/2$, $A_2 = -1/2$ and $\vartheta = 1$. Figure 5 shows the first two terms in the expansion of $h(x, t)$ and the first term in the expansion of $u(x, t)$ for the same body shape and parameter values with initial conditions (b). Notice how in both cases the free surface is forced down under the centre of the body and up at its sides, potentially causing a cushion of air to be caught between the body and the water. Care should be taken in interpreting these results since, of course, they are only a good approximation to the behaviour of the solution during the initial stages of the motion when $t \ll 1$.

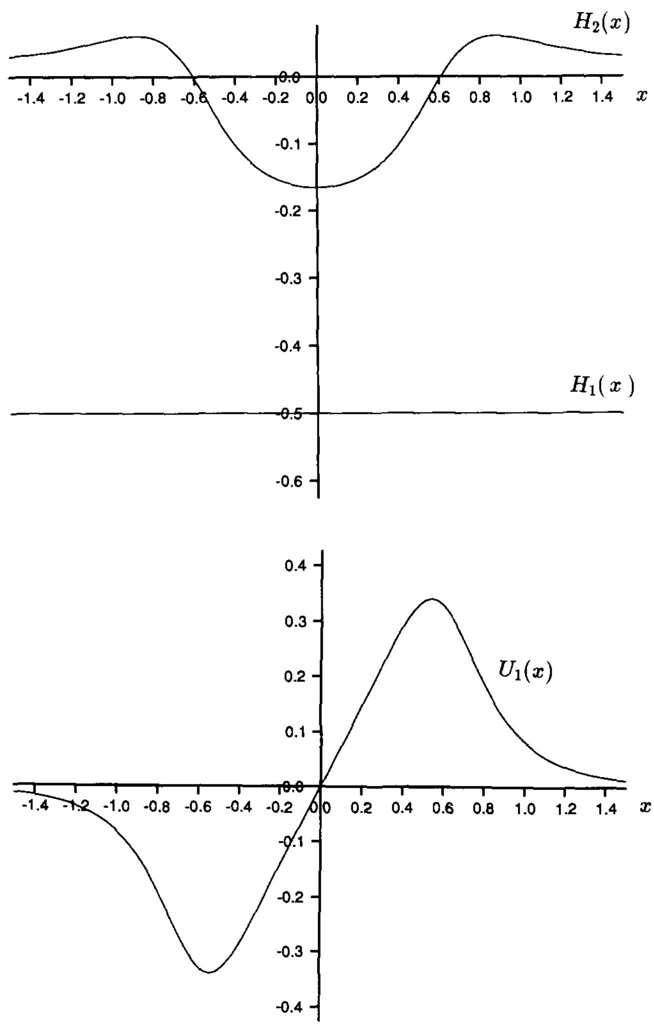

Fig. 4. The coefficients of the first two terms in the small time expansion of the free surface elevation and the first term of the horizontal air velocity for a body with profile $y = 3x^6$, subject to the initial conditions (a). The function $a(t)$ is such that $A_0 = 1$, $A_1 = -1/2$, $A_2 = -1/2$ and $\vartheta = 1$.

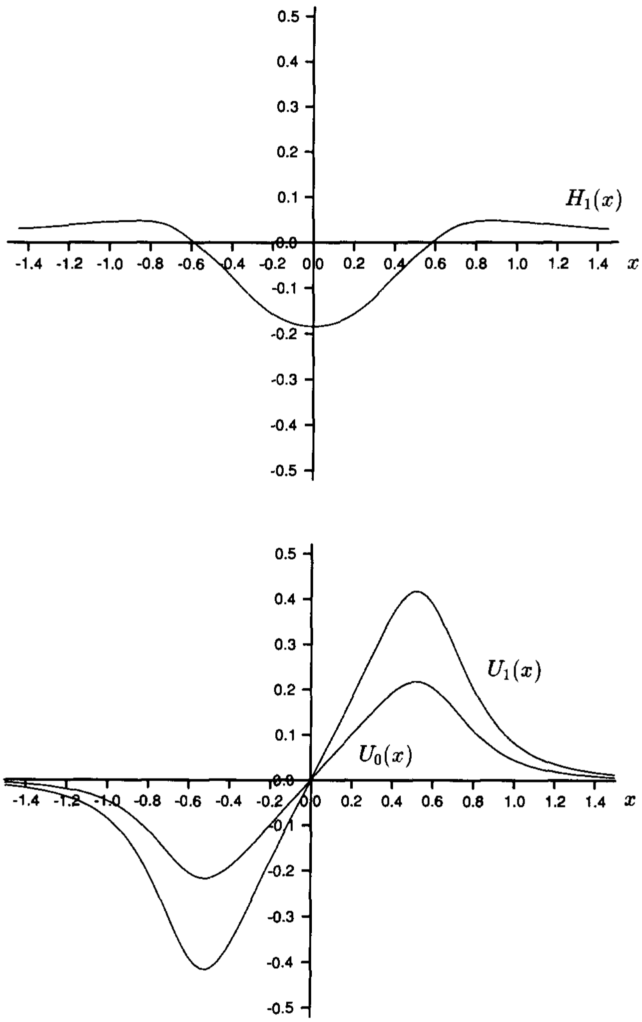

Fig. 5. The coefficients of the first term in the small time expansion of the free surface elevation and the first two terms of the horizontal air velocity for a body with profile $y = 3x^6$, subject to the initial conditions (b). The function $a(t)$ is such that $A_0 = 1$, $A_1 = -1/2$, $A_2 = -1/2$ and $\vartheta = 1$.

6.2. Weak coupling asymptotic behaviour

The pair of coupled equations (38) and (39) can also be investigated asymptotically when $\vartheta \ll 1$, corresponding to the case when the aspect ratio of the air-gap is large compared to the density ratio, and so the coupling between the air and the water problems is weak. We seek formal solutions for $u(x, t)$ and $h(x, t)$ as power series in ϑ in the form

$$u = U_0 + \vartheta U_1 + \vartheta^2 U_2 + O(\vartheta^3), \qquad h = H_0 + \vartheta H_1 + \vartheta^2 H_2 + O(\vartheta^3).$$

Substituting into equation (38) and equating coefficients of ϑ, we obtain the leading order terms

$$O(1) \qquad \frac{\partial}{\partial t}(a(t) + f(x)) + \frac{\partial}{\partial x}[(a(t) + f(x))U_0(x, t)] = 0, \tag{50a}$$

$$O(\vartheta) \qquad \frac{\partial H_0}{\partial t}(x, t) + \frac{\partial}{\partial x}[H_0(x, t)U_0(x, t) - (a(t) + f(x))U_1(x, t)] = 0, \tag{50b}$$

and from equation (39)

$$O(1) \qquad \mathscr{H}\left(\frac{\partial^2 H_0}{\partial t^2}\right) = \frac{\partial U_0}{\partial t}(x, t) + U_0(x, t)\frac{\partial U_0}{\partial x}(x, t), \tag{51a}$$

$$O(\vartheta) \qquad \mathscr{H}\left(\frac{\partial^2 H_1}{\partial t^2}\right) = \frac{\partial U_1}{\partial t}(x, t) + \frac{\partial}{\partial x}(U_0(x, t)U_1(x, t)). \tag{51b}$$

Integrating equation (50a) once with respect to x and using the symmetry condition $u(0, t) = 0$ for all t gives an expression for the leading order velocity,

$$U_0(x, t) = -\frac{a'(t)x}{a(t) + f(x)}. \tag{52}$$

$H_0(x, t)$ can then be calculated by inverting the Hilbert transform in equation (51a), yielding

$$\frac{\partial^2 H_0}{\partial t^2} = \frac{1}{\pi}\int_{-\infty}^{+\infty}\left[\frac{\xi}{(a(t) + f(\xi))^2}\{(a(t) + f(\xi))a''(t) - a'(t)^2\} - \frac{a'(t)^2\xi}{(a(t) + f(\xi))^3}\{a(t) + f(\xi) - \xi f'(\xi)\}\right]\frac{d\xi}{\xi - x}. \tag{53}$$

We need to impose two initial conditions on $H_0(x, t)$, and for simplicity we choose

$$h(x, 0) = 0 \quad \text{and} \quad \frac{\partial h}{\partial t}(x, 0) = 0$$

at $t = 0$. In general, equation (53) must be evaluated numerically. We can, however, obtain a closed form solution in the special case when the body shape is a *finite flat plate*, so that $f(x) \equiv 0$ for $|x| \leq 1$. From equation (52) we obtain $U_0(x, t) = -a'(t)x/a(t)$, and hence $V_0(x, t) = a'(t)y/a(t)$ in $|x| \leq 1$. If we insist that there is zero pressure at the ends of the air-gap, $|x| = 1$, then the corresponding pressure distribution on $y = 0$ is

$$P_0(x, t) = c''(t)(1 - x^2) \quad \text{for} \quad |x| \leq 1, \tag{54}$$

where

$$c(t) = \int_0^t \int_0^{t_1} \frac{2a'(t_2)^2 - a(t_2)a''(t_2)}{2a(t_2)^2}\, dt_2\, dt_1,$$

as shown in Fig. 6. Evaluating the integral in equation (53) gives an explicit formula for the leading order shape of the free surface, namely

$$H_0(x, t) = -\frac{2c(t)}{\pi}\left[2 + x\log\left|\frac{1 - x}{1 + x}\right|\right], \tag{55}$$

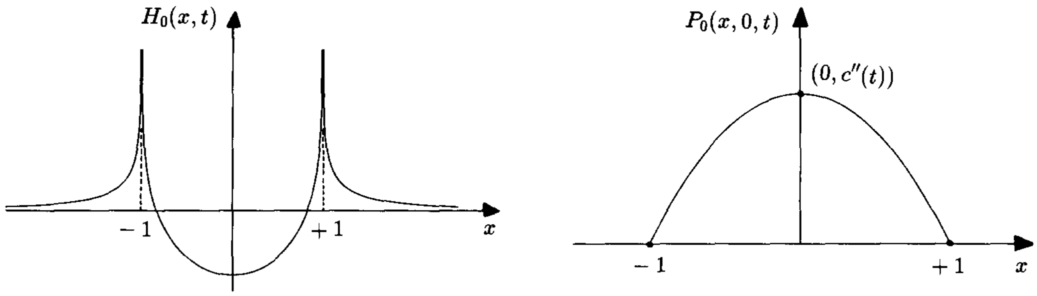

Fig. 6. Leading order pressure and free surface elevation for a flat plate of finite length in the limiting case $\vartheta = 0$.

which is also plotted in Fig. 6. Evidently $H_0(x, t) \to \infty$ as $|x| \to 1$, and the discontinuity in the pressure gradient at the edges of the plate gives rise to an infinite free surface elevation at $|x| = 1$. This limiting case (the weak coupling means that the water surface has been treated as a flat, rigid boundary in the air problem) has appeared in the papers by Verhagen [15], Lewison [12] and Asryan [1] and is the crudest possible model of air entrapment.

7. Numerical calculations

We now turn our attention to a numerical investigation of equations (38) and (39) without restriction on ϑ or t. For simplicity we impose the initial conditions

$$h(x, 0) = 0 \quad \text{and} \quad \frac{\partial h}{\partial t}(x, 0) = 0$$

at $t = 0$. Integrating equation (38) with respect to x gives an expression for $u(x, t)$ in terms of the unknown function $h(x, t)$,

$$u(x, t) = -\frac{1}{\vartheta h(x, t) - (a(t) + f(x))} \frac{\partial}{\partial t} \left[\int_0^x \{\vartheta h(\xi, t) - (a(t) + f(\xi))\} \, \mathrm{d}\xi \right],$$

which can be written more conveniently as

$$u(x, t) = \frac{1}{\vartheta h(x, t) - (a(t) + f(x))} \left[a'(t)x - \vartheta \frac{\partial F}{\partial t}(x, t) \right], \tag{56}$$

where

$$F(x, t) = \int_0^x h(\xi, t) \, \mathrm{d}\xi \, .$$

The free surface elevation is now determined from equation (39). Inverting the Hilbert transform gives

$$\frac{\partial^2 h}{\partial t^2} = -\frac{1}{\pi} \int_0^x \left[\frac{\partial u}{\partial t}(\xi, t) + u(\xi, t) \frac{\partial u}{\partial x}(\xi, t) \right] \frac{\mathrm{d}\xi}{\xi - x} \, . \tag{57}$$

The numerical procedure employed to solve equations (56) and (57) is again an iterative one, beginning with an initial guess for the free surface elevation $h(x, t)$, which is expressed as a sequence of values at n equally spaced points, $x_1, x_2, \ldots, x_n$. Using these values the function $F(x, t)$ can be evaluated at each of the points x_i for $i = 1, 2, \ldots, n$ by using one of the standard techniques for numerical quadrature. Next, the corresponding values of $u(x_i, t)$ are obtained by using equation (56) and performing a numerical differentiation with respect to t, resulting in the values of the first iteration for the horizontal air velocity. These values are then used to evaluate the singular integral in equation (57). Particular care is taken to split up the range of integration in order to remove the weak logarithmic singularity at the point $\xi = x$; the contribution from the neighbourhood of that point being approximated by expanding the integrand in a Taylor series and evaluating the resulting Cauchy Principal Value integral analytically. Finally, the result of this quadrature is integrated twice with respect to time by making use of the well-known formula,

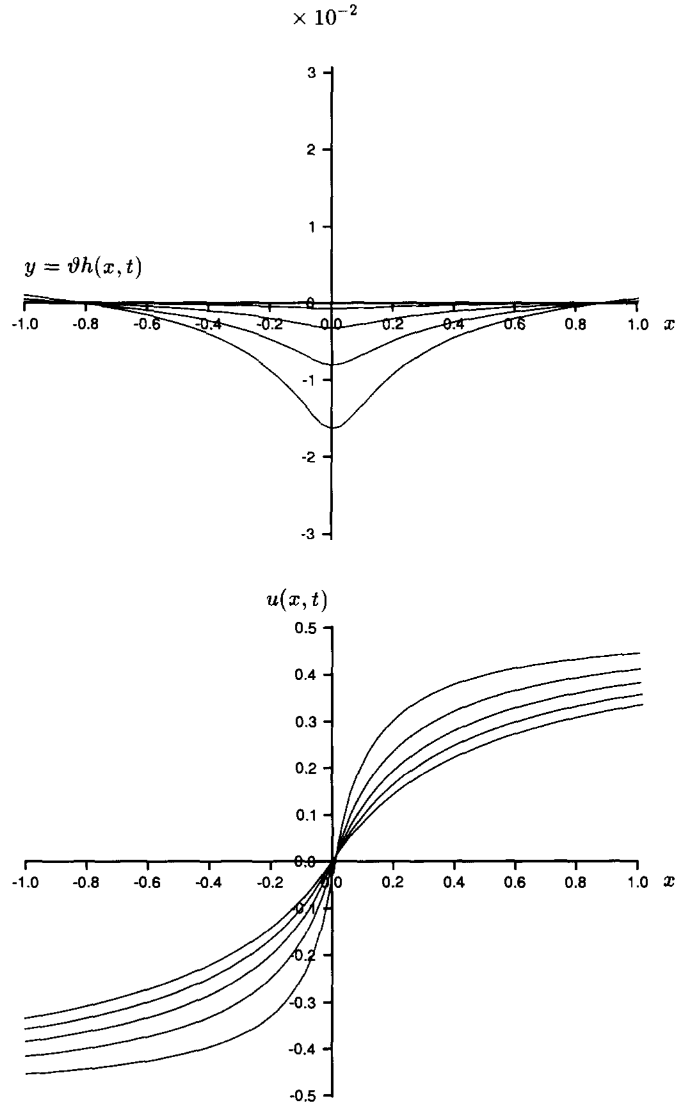

Fig. 7. Computed free surface elevation and horizontal air velocity at time $t = 0.0, 0.2, 0.4, 0.6$ and 0.8 for a wedge shaped body $y = 2|x|$ moving with constant speed, with $a(t) = 1 - t$ and $\vartheta = 0.1$. The vertical scale is exaggerated and so the body does not appear.

$$\int_0^t \int_0^{t'} g(t'')\,\mathrm{d}t''\,\mathrm{d}t' = \int_0^t (t-\xi)g(\xi)\,\mathrm{d}\xi\ ,$$

and the next iteration for $h(x, t)$ obtained. The process is repeated until a suitable criterion for convergence is satisfied.

Figure 7 shows the computed values of the horizontal air velocity and free surface elevation for a wedge shaped body, $y = 2|x|$, approaching the water with constant speed at various values of the non-dimensional time t, with the function $a(t)$ defined so that $a(t) = 1 - t$. Figures 8 and 9 show the same quantities for bodies with profiles $y = 3x^2$ and $y = 3x^6$ respectively. In all the calculations $\vartheta = 0.1$.

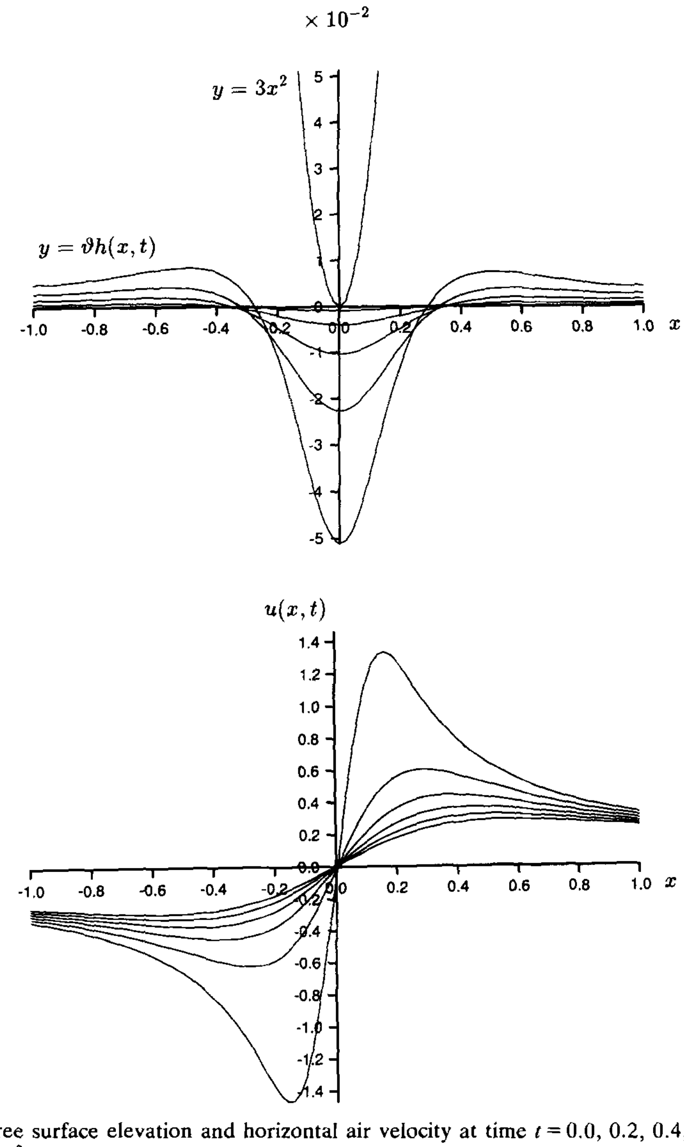

Fig. 8. Computed free surface elevation and horizontal air velocity at time $t = 0.0$, 0.2, 0.4, 0.6, 0.8 and 1.0 for a parabolic body $y = 3x^2$ moving with constant speed, with $a(t) = 1 - t$ and $\vartheta = 0.1$. The position of the body is also shown but, because of the exaggerated vertical scale, only appears at $t = 1.0$.

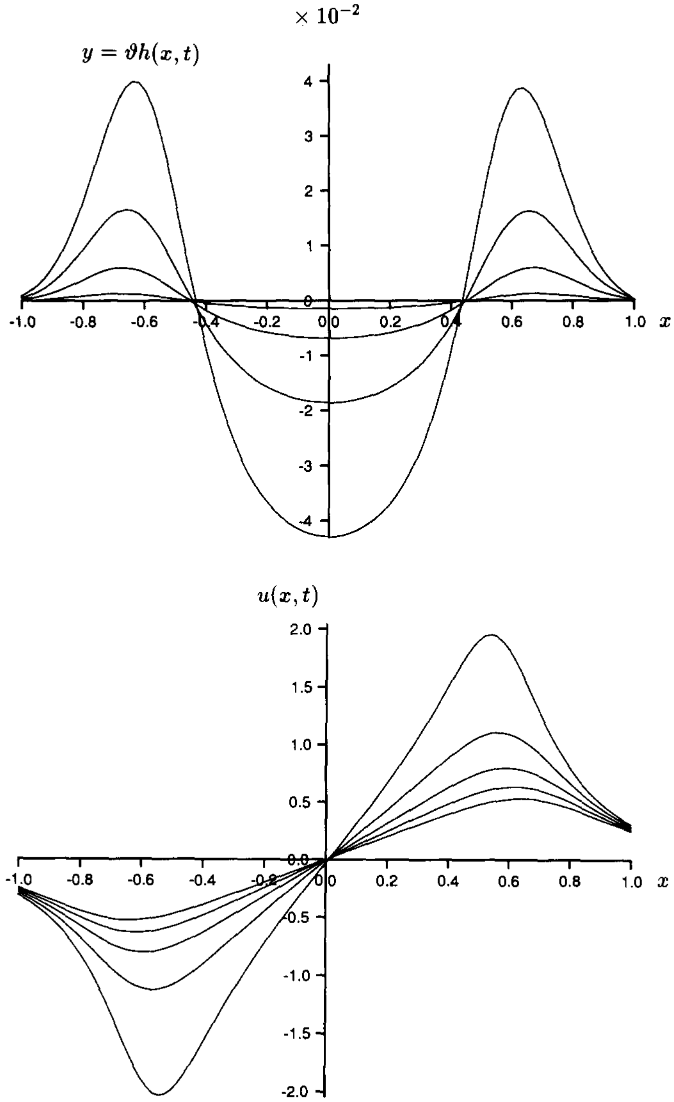

Fig. 9. Computed free surface elevation and horizontal air velocity at time $t = 0.0$, 0.2, 0.4, 0.6 and 0.8 for a body with profile $y = 3x^6$ moving with constant speed, with $a(t) = 1 - t$ and $\vartheta = 0.1$. The vertical scale is exaggerated and so the body does not appear.

We observe that the 'blunter' bodies, like the $y = 3x^6$ profile, tend to carry a body of air ahead of them as they descend. This depresses the free surface ahead of them while causing it to rise at their edges, with the consequence that they will tend to trap a pocket of air between themselves and the water at the moment of impact. In contrast the 'sharper' bodies, like the wedge, allow the air to escape more efficiently and, while still depressing the free surface ahead of them, seem unlikely to trap a significant cushion of air. These calculations can be continued until the water surface and the body touch for the first time, but as the air-gap narrows the horizontal air velocity, given by equation (56), tends to infinity. This unbounded growth is physically unacceptable since when the air speed becomes comparable with the sound seed in the air the neglected effect of air compressibility becomes significant, invalidating the incompressible model and, of course, leads to numerical problems too.

8. Comparison with earlier work

The present work justifies most of the approximations made by Verhagen [15], Lewison [12] and Asryan [1] during the initial stages of the motion, when compressibility effects in the air are negligible. However, no justification has been found for the first two authors' modelling of the effect of the compressibility of the air, for which they employed the ideas of classical steady nozzle flow to an intrinsically unsteady problem, and their results must therefore be treated with caution. Asryan's [1] calculations apply only to the earliest stages of the motion before the deflection of the free surface becomes comparable with the thickness of the air gap, and therefore his estimates for the time of impact are unlikely to be accurate.

Direct experimental observations of the thin air gap and the free surface during the impact are difficult, and the author is unaware of any quantitative experimental data for the air velocity and free surface deformation. However, the pressure histories presented by Driscoll and Lloyd [5] for flat-bottomed wedges show that the first contact between the body and the water is usually made at the junction of the keel and the sloping face of the wedge, suggesting that even the simplest model, leading to equations (54) and (55), describes the basic process.

9. Conclusions and further work

The present incompressible leading order theory for the air layer between a moving solid body and a free surface of a fluid predicts that some shapes of body may trap a pocket of air as they impact onto the free surface. However, no satisfactory method for predicting the size of the entrapped air bubble has been obtained.

The model has a number of limitations. Most seriously, it is limited to incompressible 'air' and so is invalid when the air speed in the narrowing gap approaches the sound speed, and hence cannot be used to predict the size of the entrapped air pocket. The models presented by other authors which attempt to represent the effects of compressibility are unsatisfactory, and a compressible version of the present theory is required. Unfortunately, the compressibility effects occur just as the free surface effects are most significant and the problem is therefore a difficult one. One approach would be to attempt a direct numerical solution of the full Euler equations in the air and the water, in which case the present results form a simple check on the early stages of the calculation before compressibility becomes significant. In addition, as the gap narrows the experiments indicate that the large horizontal air velocities can cause the peaks in the free surface to break up into spray. The phenomenon may have a significant effect on the size of the trapped air pocket and needs to be investigated.

We have limited the present discussion to the case of a body approaching a free surface, but the same model is obviously still applicable if the body is moving away. Although of less practical importance, this situation should also be investigated.

Finally, we note that after impact has occurred the experimental results of Verhagen [15] and Lewison [12] indicate that the trapped air cushion breaks up into bubbles at its edges, which expand into the centre of the pocket at a speed comparable to the sound speed. It may be possible to model these pressure oscillations of the trapped air bubble using the classical theory of shocks in one-dimensional unsteady gas dynamics.

Acknowledgements

The author wishes to thank Dr J.R. Ockendon (University of Oxford) for his invaluable assistance and for acting as his D. Phil. supervisor while most of the present research was performed, and Professor N. Riley (University of East Anglia) for a number of helpful comments and suggestions. The work was funded jointly by S.E.R.C. and the Admiralty through the C.A.S.E. Studentship Scheme.

References

1. N.G. Asryan, Solid plate impact on surface of incompressible fluid in the presence of a gas layer between them. *Izv. Akad. Nauk Arm. SSR Mekh.* 25 (1972) 32–49 (in Russian).
2. S-L. Chuang, Experiments on slamming of wedge shaped bodies. *J. Ship Res.* 11 (3) (1967) 190–198.
3. R. Cointe and J-L. Armand, Hydrodynamic impact analysis of a cylinder. *A.S.M.E. J. Offshore Mech. Arc. Eng.* 109 (1987) 237–243.
4. R. Cointe, Two-dimensional water-solid impact. *A.S.M.E. J. Offshore Mech. Arc. Eng.* 111 (1989) 109–114.
5. A. Driscoll and A. Lloyd, Slamming Experiments – Description of Facilities and Details of Impact Pressure Results. *Report A.M.T.E. (H) R82002* (1982) 53 pp AMTE (Haslar), Haslar, Gosport, Hants.
6. V.A. Eroshin, A.V. Plyusnin, N.I. Romanenkov, Yu.A. Sozonenko and Yu.L. Yakimov, Atmosphere influence on hydrodynamic forces at a plane disc impact on a compressible fluid surface. *Izv. Akad. Nauk SSSR Mekh. Zhidk. Gaza* 3 (1984) 15–20 (in Russian).
7. M. Greenhow, Wedge entry into initially calm water. *Appl. Ocean Res.* 9 (4) (1987) 214–223.
8. K. Hagiwara and T. Yuhara, Fundamental study of wave impact load on a ship bow. *J. Soc. Nav. Arch. Japan* 135 (1974) 181–189.
9. S.D. Howison, J.R. Ockendon and S.K. Wilson, Incompressible water entry problems at small deadrise angles. *J. Fluid Mech.* 222 (1991) 215–230.
10. A.A. Korobkin and V.V. Pukhnachov, Initial stage of water impact. *Ann. Rev. Fluid Mech.* 20 (1988) 159–185.
11. G.R.G. Lewison and W.M. Maclean, On the cushioning of water impact by entrapped air. *J. Ship Res.* 12 (1968) 116–130.
12. G.R.G. Lewison, On the reduction of slamming pressures. *Trans. R.I.N.A.* 112 (1970) 285–306.
13. J.P. Moran, On the hydrodynamic theory of water-exit and -entry. *TAR-TR 6501* (1965) 124 pp. Therm Advanced Research Inc., Ithaca, New York.
14. W.C.E. Nethercote, M. Mackay and B. Menon, Some warship slamming investigations. *D.R.E.A. Technical Memorandum 86/206* (1986) 34 pp.
15. J.H.G. Verhagen, The impact of a flat plate on a water surface. *J. Ship Res.* 11 (1967) 211–223.
16. T. Von Kármán, The impact on seaplane floats during landing. *N.A.C.A. T.N. No. 321* (1929) 9 pp.
17. H. Wagner, Über Stoß- und Gleitvorgänge an der Oberfläche von Flüssigkeiten. *Z.A.M.M.* 12 (4) (1931) 193–215 (in German). Translated into English as Phenomena associated with impact and sliding on liquid surfaces. *N.A.C.A. Translation 1366.* 60 pp.